# Foundations of Probabilistic Proofs

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## Lecture 02

# Sumcheck Protocol

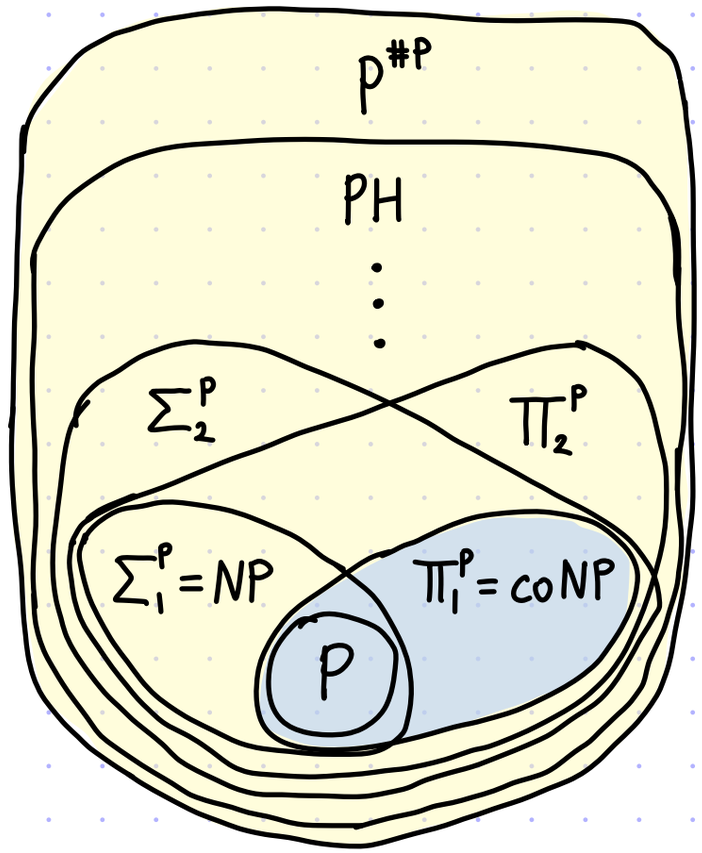
# Interactive Proofs for Counting Problems

We saw an IP for GNI, a problem in coNP not known to be in P.
But GNI is not believed to be coNP-complete. [If so, PH collapses to 2nd level.]
Today we prove:

**theorem:** UNSAT $\in$ IP, so coNP $\subseteq$ IP

**theorem:** #SAT $\in$ IP, so $P^{\#P} \subseteq$ IP

$P^{\#P}$: languages decidable in polynomial time via a machine with a #SAT oracle




These results are surprising:

- Many languages beyond NP!
- The IP for GNI uses properties of graph isomorphisms, but UNSAT and #SAT do not seem to have similar properties.

We learn new ideas: ARITHMETIZATION, SUMCHECK PROTOCOL.

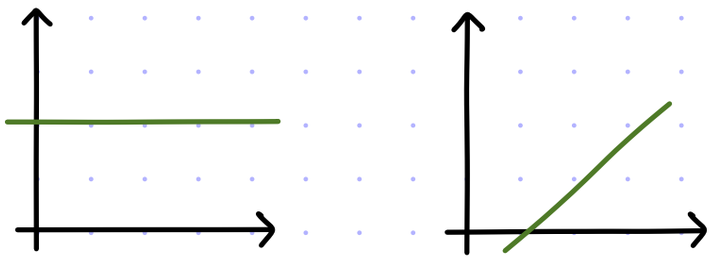
# Preliminaries: Zeros of Univariate Polynomials

Basic question: how many zeros can a univariate polynomial $p$ have?

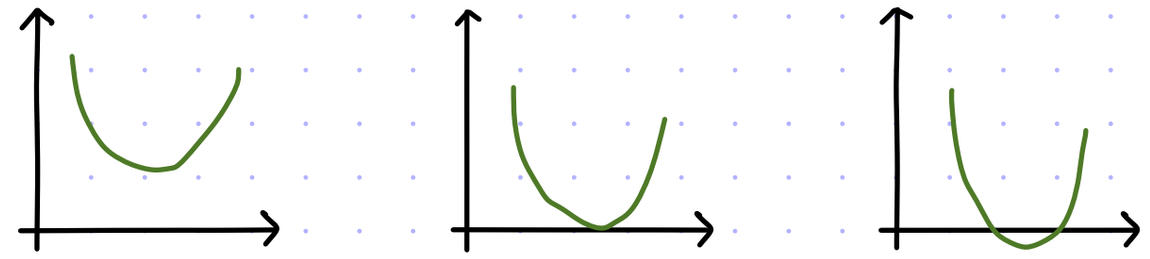
If $p \equiv 0$ then lots.

So assume that $p \not\equiv 0$. In this case the answer depends on the degree of $p$.

Ex: if $p$ has degree 1
then $p$ has $\leq 1$ zeros



Ex: if $p$ has degree 2
then $p$ has $\leq 2$ zeros



In general: $p \in \mathbb{F}[x]$ has at most $\deg(p)$ zeros in $\mathbb{F}$
(and exactly $\deg(p)$ zeros in the algebraic closure of $\mathbb{F}$)

This directly leads to an invaluable fact:

Polynomial Identity Lemma: $\forall$ non-zero $f \in \mathbb{F}[x]$ $\forall S \subseteq \mathbb{F}$, $\Pr_{\alpha \leftarrow S}[f(\alpha) = 0] \leq \frac{\deg(f)}{|S|}$

Hence, $\forall f, g \in \mathbb{F}[x]$ if $f \neq g$ then $\Pr_{\alpha \leftarrow S}[f(\alpha) = g(\alpha)] \leq \frac{\max\{\deg(f), \deg(g)\}}{|S|}$.

# Preliminaries: Zeros of Multivariate Polynomials

Basic question: how many zeros can a multivariate polynomial $p$ have?

If $p \equiv 0$ then lots.

So assume that $p \not\equiv 0$. In this case the answer depends on the degree of $p$.

But what do we mean by "degree" of a polynomial $p(x_1,\ldots,x_n) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n}$ ?

- individual degree: $\deg_{ind}(p) := \max\{ \max\{i_1,\ldots,i_n\} \mid (i_1,\ldots,i_n) \text{ s.t. } c_{i_1,\ldots,i_n} \neq 0 \}$.
- total degree: $\deg_{tot}(p) := \max\{ i_1+\cdots+i_n \mid (i_1,\ldots,i_n) \text{ s.t. } c_{i_1,\ldots,i_n} \neq 0 \}$.

Examples:
- $p_1 = x_1x_2 + x_1x_4 + x_3$, $\deg_{ind}(p_1) = 1$, $\deg_{tot}(p_1) = 2$
- $p_2 = x_1^3 + x_1x_2 + x_3x_4^2$, $\deg_{ind}(p_2) = 3$, $\deg_{tot}(p_2) = 3$

In general, $\deg_{ind}(p) \le \deg_{tot}(p) \le n \cdot \deg_{ind}(p)$. (Both bounds can be tight.)

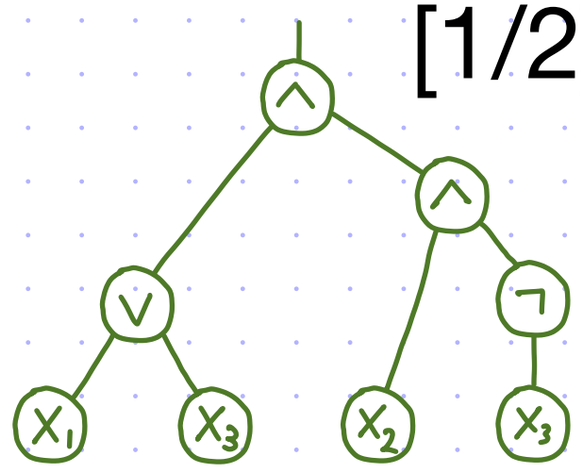
The PIL extends to multivariate polynomials (by induction on $n$):

$$\forall \text{ non-zero } f \in \mathbb{F}[x_1,\ldots,x_n] \quad \forall S \subseteq \mathbb{F}, \quad \Pr_{\alpha_1,\ldots,\alpha_n \leftarrow S}[f(\alpha_1,\ldots,\alpha_n) = 0] \le \frac{\deg_{tot}(f)}{|S|}.$$

There are refinements and generalizations.

The bound can be tight: $\Pr_{\alpha_1,\ldots,\alpha_n \leftarrow \mathbb{F}}[\prod_{j=1}^{d}(\alpha_1 - \gamma_j) = 0] = \frac{d}{|\mathbb{F}|}$ (for distinct $\gamma_1,\ldots,\gamma_d \in \mathbb{F}$), $\Pr_{\alpha_1,\ldots,\alpha_n \leftarrow \mathbb{F}}[\prod_{i=1}^{n}\alpha_i = 0] = 1-(1-\frac{1}{|\mathbb{F}|})^n \ge \frac{n}{|\mathbb{F}|} - \frac{1}{2}\cdot\frac{n^2}{|\mathbb{F}|^2}$, ...

# Arithmetization of a Boolean Formula [1/2]



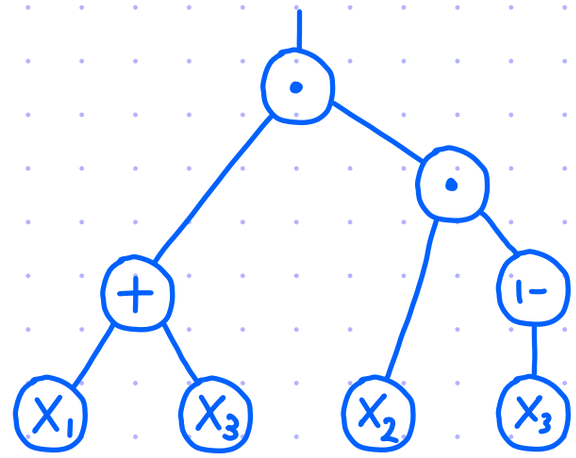

A boolean formula $\varphi(x_1,\dots,x_n)$ is a tree where:

- every leaf vertex is labeled with a variable $x_i$;
- every internal vertex is a logical operator on its children.

$\neg, \wedge, \vee$

Arithmetization replaces each logical operator with an arithmetic operator:

$$\begin{aligned} \neg x &\mapsto 1-x \\ x \wedge y &\mapsto x \cdot y \\ x \vee y &\mapsto x+y \end{aligned}$$




$(x_1+x_3)\cdot(x_2\cdot(1-x_3))$

$= x_1x_2 - x_1x_2x_3 + x_2x_3 - x_2x_3^2$

We obtain an expression for a polynomial $p(x_1,\dots,x_n)$ s.t.

- $\deg_{tot}(p) \le$ # leaves in $\varphi$
  $\le$ # vertices in $\varphi$
  $= |\varphi|$

$\deg_{tot}(x_i) = 1$
$\deg_{tot}(1-p) \le \deg_{tot}(p)$
$\deg_{tot}(p_1+p_2) \le \max\{\deg_{tot}(p_1), \deg_{tot}(p_2)\}$
$\deg_{tot}(p_1\cdot p_2) \le \deg_{tot}(p_1)+\deg_{tot}(p_2)$

- can evaluate the expression for $p$ in $\le |\varphi|$ arithmetic operations:

  # internal vertices in $\varphi$ $\le$ # vertices in $\varphi$ $= |\varphi|$

But what does $p$ have to do with $\varphi$? Not much in general.

But for 3CNFs we get something useful!

(Also for $\varphi$ where every NOT is on an input.)

# Arithmetization of a Boolean Formula [2/2]

We focus on the case of a 3CNF with $m$ clauses. 3CNF = conjunctive normal form with 3 literals per clause

It is specified by a subset $S \subseteq [n]^3 \times \{0,1\}^3$ with $|S| = m$:

$$\varphi(x_1,\dots,x_n) = \bigwedge_{\substack{(j_1,j_2,j_3,\\ b_1,b_2,b_3)\in S}} \left(\mathrm{neg}(b_1,x_{j_1}) \vee \mathrm{neg}(b_2,x_{j_2}) \vee \mathrm{neg}(b_3,x_{j_3})\right) .$$

Example: $S = \{(1,3,4,0,1,1),(1,2,5,0,1,0),(3,4,5,1,0,1)\} \mapsto (x_1 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_2 \vee x_5) \wedge (\bar{x}_3 \vee x_4 \vee \bar{x}_5)$ .

In this case the expression for the polynomial is:

$$p(x_1,\dots,x_n) = \prod_{\substack{(j_1,j_2,j_3,\\ b_1,b_2,b_3)\in S}} \left(\mathrm{neg}(b_1,x_{j_1}) + \mathrm{neg}(b_2,x_{j_2}) + \mathrm{neg}(b_3,x_{j_3})\right) .$$

claim:

- $\varphi \in$ UNSAT $\rightarrow \sum_{a_1,\dots,a_n\in\{0,1\}} p(a_1,\dots,a_n) = 0$
- $\varphi \notin$ UNSAT $\rightarrow 0 < \sum_{a_1,\dots,a_n\in\{0,1\}} p(a_1,\dots,a_n) \leq 2^n \cdot 3^m$

corollary: $\forall$ prime $q > 2^n \cdot 3^m$

$$\varphi \in \text{UNSAT} \leftrightarrow \sum_{a_1,\dots,a_n\in\{0,1\}} p(a_1,\dots,a_n) = 0 \mod q$$

# Sumcheck Protocol [recursive description]

A protocol for polynomial summations of the form $\sum_{\alpha_1,\ldots,\alpha_n \in H} p(\alpha_1,\ldots,\alpha_n) = \gamma$.

$\deg_{\text{ind}}(p) \le d$

| | $P(\mathbb{F}, H, n, \gamma, p)$ | | $V^p(\mathbb{F}, H, n, \gamma, d)$ |
|---|---|---|---|
| Case $n=0$: | Do nothing. | | Check that $p = \gamma$. |
| Case $n>0$: | $p_1(X) := \sum_{\alpha_2,\ldots,\alpha_n \in H} p(X, \alpha_2, \ldots, \alpha_n)$ | $\xrightarrow{p_1 \in \mathbb{F}[X]}$ | $\sum_{\alpha_1 \in H} p_1(\alpha_1) \stackrel{?}{=} \gamma$ |
| | | $\xleftarrow{\omega_1 \in \mathbb{F}}$ | $\omega_1 \leftarrow \mathbb{F}$ |

Set $p'(X_2,\ldots,X_n) := p(\omega_1, X_2, \ldots, X_n)$ and $\gamma' := p_1(\omega_1)$.

$P(\mathbb{F}, H, n-1, \gamma', p')$ $\quad \sum_{\alpha_2,\ldots,\alpha_n \in H} p'(\alpha_2,\ldots,\alpha_n) = \gamma'$ $\quad V^{p'}(\mathbb{F}, H, n-1, \gamma', d)$

**Completeness:** $\sum_{\alpha_1,\ldots,\alpha_n} p(\alpha_1,\ldots,\alpha_n) = \gamma \rightarrow \begin{cases} \text{if } n=0 \text{ then } p = \gamma \\ \text{if } n>0 \text{ then } \sum_{\alpha_2,\ldots,\alpha_n} p'(\alpha_2,\ldots,\alpha_n) = \sum_{\alpha_2,\ldots,\alpha_n} p(\omega_1, \alpha_2, \ldots, \alpha_n) = p_1(\omega_1) = \gamma' \end{cases}$

**Note:** $V^p$ queries $p$ at $(\omega_1,\ldots,\omega_n)$ (and makes no other queries).

# Sumcheck Protocol [iterative description]

By "unrolling" the recursion we obtain an iterative description of the protocol.

$P(\mathbb{F}, H, n, \gamma, p)$ $\sum_{\alpha_1,\dots,\alpha_n \in H} p(\alpha_1,\dots,\alpha_n) \stackrel{?}{=} \gamma$ $V^p(\mathbb{F}, H, n, \gamma, d)$ ← $\deg_{ind}(p) \le d$

$p_1(X) := \sum_{\alpha_2,\dots,\alpha_n \in H} p(X, \alpha_2, \dots, \alpha_n)$ — $p_1 \in \mathbb{F}[x]$ → — $\sum_{\alpha_1 \in H} p_1(\alpha_1) \stackrel{?}{=} \gamma$

← $\omega_1 \in \mathbb{F}$ — $\omega_1 \leftarrow \mathbb{F}$

$p_2(X) := \sum_{\alpha_3,\dots,\alpha_n \in H} p(\omega_1, X, \alpha_3, \dots, \alpha_n)$ — $p_2 \in \mathbb{F}[x]$ → — $\sum_{\alpha_2 \in H} p_2(\alpha_2) \stackrel{?}{=} p_1(\omega_1)$

← $\omega_2 \in \mathbb{F}$ — $\omega_2 \leftarrow \mathbb{F}$

⋮ ⋮ ⋮

$p_n(X) := p(\omega_1, \dots, \omega_{n-1}, X)$ — $p_n \in \mathbb{F}[x]$ → — $\sum_{\alpha_n \in H} p_n(\alpha_n) \stackrel{?}{=} p_{n-1}(\omega_{n-1})$

← $\omega_n \in \mathbb{F}$ — $\omega_n \leftarrow \mathbb{F}$

$p(\omega_1, \dots, \omega_n) \stackrel{?}{=} p_n(\omega_n)$

$O(n \cdot d)$ elements

$O(n \cdot |H| \cdot d)$ field ops + 1 eval of $p$

# Soundness of Sumcheck Protocol [1/2]

**claim:** $\sum_{\alpha_1,\ldots,\alpha_n \in H} p(\alpha_1,\ldots,\alpha_n) \neq \gamma \longrightarrow \forall \tilde{P}\ \Pr[\langle \tilde{P}, V^p(\mathbb{F},H,n,\gamma,d)\rangle = 1] \leq 1-(1-\frac{d}{|\mathbb{F}|})^n \leq \frac{nd}{|\mathbb{F}|}$.

More generally the bound is $1-\prod_{i\in[n]}(1-\frac{d_i}{|S_i|})$ if $\deg_{x_i}(p)\leq d_i$ and $w_i$ is sampled from $S_i \subseteq \mathbb{F}$.

**proof:** The malicious prover $\tilde{P}$ is described by $n$ polynomials $\tilde{p}_1,\ldots,\tilde{p}_n \in \mathbb{F}^{\leq d}[X]$ where $\tilde{p}_i$ depends on the verifier messages $w_1,\ldots,w_{i-1} \in \mathbb{F}$.

The proof is by induction on $n$.

- Base case: $n=1$. We show that $\Pr[\langle \tilde{P}, V^p(\mathbb{F},H,n=1,\gamma,d)\rangle = 1] \leq 1-(1-\frac{d}{|\mathbb{F}|}) = \frac{d}{|\mathbb{F}|}$.

$P(\mathbb{F},H,n=1,\gamma,p)$ | $\sum_{\alpha_1\in H} p(\alpha_1) \stackrel{?}{=} \gamma$ | $V^p(\mathbb{F},H,n=1,\gamma,d)$

$p_1(X) := p(X)$ | $p_1 \in \mathbb{F}[X]$ → | $\sum_{\alpha_1\in H} p_1(\alpha_1) \stackrel{?}{=} \gamma$

← $w_1 \in \mathbb{F}$ | $w_1 \leftarrow \mathbb{F}$

| $p(w_1) \stackrel{?}{=} p_1(w_1)$

Assume that $\sum_{\alpha_1\in H} \tilde{p}_1(\alpha_1) = \gamma$, since otherwise the verifier immediately rejects.

Hence $\tilde{p}_1 \neq p_1$ because $\sum_{\alpha_1\in H} p_1(\alpha_1) = \sum_{\alpha_1\in H} p(\alpha_1) \neq \gamma$.

We conclude that $\Pr[\langle \tilde{P}, V^p(\mathbb{F},H,n=1,\gamma,d)\rangle = 1] = \Pr[p(w_1) = \tilde{p}_1(w_1)] = \Pr[p_1(w_1) = \tilde{p}_1(w_1)] \leq \frac{d}{|\mathbb{F}|}$.

# Soundness of Sumcheck Protocol [2/2]

claim: $\sum_{\alpha_1,\dots,\alpha_n \in H} p(\alpha_1,\dots,\alpha_n) \neq \gamma \longrightarrow \forall\, \tilde{P}\ \Pr[\langle \tilde{P}, V^p(\mathbb{F},H,n,\gamma,d)\rangle = 1] \leq 1-(1-\frac{d}{|\mathbb{F}|})^n \leq \frac{nd}{|\mathbb{F}|}$.

proof: [continued]

- Inductive case: $n>1$. Assume the claim for $(n-1)$-variate polynomials.

We can assume that $\sum_{\alpha_1 \in H} \tilde{p}_1(\alpha_1) = \gamma$ since otherwise the verifier immediately rejects.

Hence $\tilde{p}_1 \neq p_1$ because $\sum_{\alpha_1 \in H} p_1(\alpha_1) = \sum_{\alpha_1,\dots,\alpha_n \in H} p(\alpha_1,\dots,\alpha_n) \neq \gamma$.

Define $E_1 := "\tilde{p}_1(w_1) \neq \sum_{\alpha_2,\dots,\alpha_n \in H} p(w_1,\alpha_2,\dots,\alpha_n)"$.

Then $\Pr[E_1] = \Pr_{w_1 \leftarrow \mathbb{F}}[\tilde{p}_1(w_1) \neq p_1(w_1)] \geq 1-\frac{d}{|\mathbb{F}|}$.

Define $E_2 := "$ the verifier rejects in round $i \geq 2"$.

By the induction hypothesis, $\Pr[E_2 | E_1] \geq (1-\frac{d}{|\mathbb{F}|})^{n-1}$.

We conclude that $\Pr[\langle \tilde{P}, V^p(\mathbb{F},H,n,\gamma,d)\rangle = 1]$

$= 1 - \Pr[E_2]$

$\leq 1 - \Pr[E_2|E_1]\cdot\Pr[E_1]$

$\leq 1 - (1-\frac{d}{|\mathbb{F}|})^{n-1}\cdot(1-\frac{d}{|\mathbb{F}|}) = 1-(1-\frac{d}{|\mathbb{F}|})^n$.

■

# Interactive Proof for UNSAT

We describe an IP for $\text{UNSAT} = \{\varphi \mid \varphi \text{ is an unsatisfiable 3CNF boolean formula}\}$.

This implies that $\text{coNP} \subseteq \text{IP}$ (by reducing to UNSAT via polynomial-time reductions).

$P(\varphi)$ — $\xrightarrow{q \in \mathbb{N}}$ — $V(\varphi)$

$2^n \cdot 3^m < q < 2^{\text{poly}(m,n)}$

$q \in \text{PRIMES}$

$p := \text{ARITH}(\varphi, \mathbb{F}_q)$ — $p := \text{ARITH}(\varphi, \mathbb{F}_q)$

$P_{sc}(\mathbb{F}_q, \{0,1\}, n, 0, p)$ — sumcheck protocol for $\sum_{\alpha_1,\dots,\alpha_n \in \{0,1\}} p(\alpha_1,\dots,\alpha_n) = 0$ — $V_{sc}^{p}(\mathbb{F}_q, \{0,1\}, n, 0, \deg_{ind}(p))$

(field, domain, #vars, sum, polynomial) — (field, domain, #vars, sum, ind degree)

$(w_1,\dots,w_n) \in \mathbb{F}_q^n \rightsquigarrow p(w_1,\dots,w_n) \in \mathbb{F}_q$ (poly(m,n) time)

**Completeness:** $\varphi \in \text{UNSAT} \to \sum_{\alpha_1,\dots,\alpha_n} p(\alpha_1,\dots,\alpha_n) = 0 \to$ P always convinces V.

**Soundness:** $\varphi \notin \text{UNSAT} \to \sum_{\alpha_1,\dots,\alpha_n} p(\alpha_1,\dots,\alpha_n) \neq 0 \to$ error is $\leq \frac{n \cdot \deg_{ind}(p)}{q} \leq \frac{n \cdot \deg_{tot}(p)}{q} \leq \frac{n \cdot m}{q} < \frac{n \cdot m}{2^n 3^m}$.

# Arithmetization for #SAT

The arithmetization we used for UNSAT was coarse:

$$\forall (a_1,\dots,a_n)\in\{0,1\}^n \qquad \varphi(a_1,\dots,a_n)=0 \;\longrightarrow\; p(a_1,\dots,a_n)=0$$
$$\varphi(a_1,\dots,a_n)=1 \;\longrightarrow\; 0<p(a_1,\dots,a_n)\leq 3^m$$

We can modify the arithmetization to be more precise:

$$\begin{aligned} \neg x &\mapsto 1-x \\ x\wedge y &\mapsto x\cdot y \\ x\vee y &\mapsto x+y-x\cdot y \end{aligned}$$

Similarly to before: $\deg_{tot}(p)\leq|\varphi|$ and can evaluate $p$ in $\leq O(|\varphi|)$ operations.

But now the value of $p$ on boolean inputs is $\varphi$:

<u>claim:</u> $\forall (a_1,\dots,a_n)\in\{0,1\}^n \qquad \varphi(a_1,\dots,a_n)=0 \;\longrightarrow\; p(a_1,\dots,a_n)=0$
$$\varphi(a_1,\dots,a_n)=1 \;\longrightarrow\; p(a_1,\dots,a_n)=1$$

$p$ is a low-degree extension of $\varphi$ because $p|_{\{0,1\}^n}\equiv\varphi$

We can now reduce #SAT to a sumcheck problem:

<u>corollary:</u> $\forall$ prime $q>2^n \qquad \#\varphi=c \longleftrightarrow \sum_{a_1,\dots,a_n\in\{0,1\}} p(a_1,\dots,a_n)=c \mod q$

# Interactive Proof for #SAT

We describe an IP for $\#SAT = \{(\varphi, c) \mid \varphi \text{ is a boolean formula with } c \text{ satisfiable assignments}\}$.

$P((\varphi, c))$

$V((\varphi, c))$

$\xrightarrow{q \in \mathbb{N}}$

$2^n < q < 2^{poly(m,n)}$

$q \in PRIMES$

$p := \text{ARITH}^{\#}(\varphi, \mathbb{F}_q)$

$p := \text{ARITH}^{\#}(\varphi, \mathbb{F}_q)$

$P_{sc}(\mathbb{F}_q, \{0,1\}, n, c, p)$
field domain #vars sum polynomial

sumcheck protocol for $\sum_{\alpha_1,\ldots,\alpha_n \in \{0,1\}} p(\alpha_1,\ldots,\alpha_n) = c$

$V_{sc}^{p}(\mathbb{F}_q, \{0,1\}, n, c, deg_{ind}(p))$
field domain #vars sum ind degree

$(w_1,\ldots,w_n) \in \mathbb{F}_q^n \rightsquigarrow p(w_1,\ldots,w_n) \in \mathbb{F}_q$
poly(m,n) time

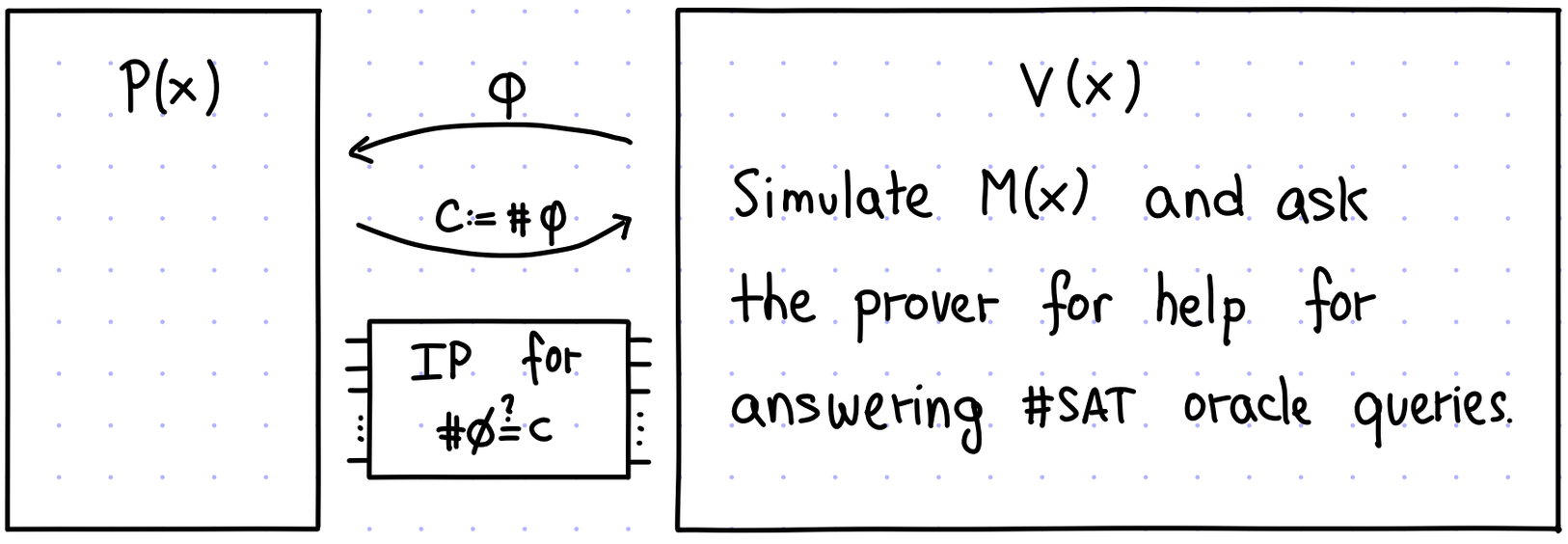
**Completeness:** $(\varphi, c) \in \#SAT \rightarrow \sum_{\alpha_1,\ldots,\alpha_n} p(\alpha_1,\ldots,\alpha_n) = c \rightarrow$ P always convinces V.

**Soundness:** $(\varphi, c) \notin \#SAT \rightarrow \sum_{\alpha_1,\ldots,\alpha_n} p(\alpha_1,\ldots,\alpha_n) \neq c \rightarrow$ error is $\leq \frac{n \cdot deg_{ind}(p)}{q} \leq \frac{n \cdot deg_{tot}(p)}{q} \leq \frac{n \cdot 3m}{q} < \frac{n \cdot 3m}{2^n}$.

# Interactive Proof for $P^{\#P}$

**theorem:** $P^{\#P} \subseteq IP$

**proof:** Let $L \in P^{\#P}$, and let $M$ be a machine that decides $L$ with a #SAT oracle. We describe an IP for $L$.




**Completeness:** $x \in L \rightarrow \Pr[\langle P(x), V(x)\rangle = 1] = 1$ because: $M^{\#SAT}(x) = 1$; $P(x)$ answers correctly each #SAT call; and the IP for #SAT has perfect completeness.

**Soundness:** $x \notin L \rightarrow \forall \tilde{P}\ \Pr[\langle \tilde{P}, V(x)\rangle = 1] \leq \varepsilon_0$ where $\varepsilon_0$ = "soundess error of the IP for #SAT". Indeed, $M^{\#SAT}(x) = 0$ so the only way for $M(x) = 1$ in the simulation is that there is a #SAT query that $\tilde{P}$ lies on, in which case we rely on the soundness of the IP for #SAT. ■

# History of Arithmetization

Arithmetization loosely refers to useful ways to map problems in boolean logic to problems that involve arithmetic.

$\{0,1\}, \Sigma \rightarrow \mathbb{N}, \mathbb{Q}, \mathbb{F}$

logical ops → arithmetic ops

- [Gödel 1931]: encode a string $(a_1,\dots,a_n) \in \Sigma^n$ as a number $p_1^{a_1} \cdots p_n^{a_n} \in \mathbb{N}$ ($PRIMES = \{p_1, p_2, \dots\}$)
  This Gödel numbering enables establishing the incompleteness of formal arithmetic.
- [Church 1936]: uses ↯ to prove that a specific problem in number theory is undecidable
  (Subsequently, Turing showed how to do this via Turing machines.)

→ Convenient injection from strings to natural numbers.

- [Razborov 1987] [Smolensky 1987]: map boolean circuit to a low-degree polynomial approximation
  This enables proving circuit complexity lower bounds. (E.g. $PARITY \notin AC_0$.)

← poly(n)-size O(1)-depth circuits with any fan-in

→ Polynomials of low degree cannot describe certain functions.

- [Lund Fortnow Karloff Nisan 1992]: $PERMANENT \in IP$ by viewing the computation of the permanent as a sumcheck claim
- [Shamir 1992]: more boolean logic → polynomial arithmetic
- ... huge role in the probabilistic proof literature

efficient encoding/decoding
multiplication property
rich automorphism group
local testing/decoding/correction
...

→ Polynomials of low degree are codes with MANY useful properties.

# Bonus: Sumcheck Protocol as a Reduction

The sumcheck protocol can be phrased as a reduction (from a polynomial summation to a single evaluation):

$$\begin{cases} \sum_{\alpha_1,\dots,\alpha_n \in H} p(\alpha_1,\dots,\alpha_n) = \gamma \longrightarrow \Pr\left[ p(w_1,\dots,w_n)=v \text{ where } ((w_1,\dots,w_n),v) \leftarrow \langle P(\mathbb{F},H,n,\gamma,p), V(\mathbb{F},H,n,\gamma,d)\rangle \right] = 1 \\ \sum_{\alpha_1,\dots,\alpha_n \in H} p(\alpha_1,\dots,\alpha_n) \neq \gamma \longrightarrow \forall \tilde{P}\ \Pr\left[ p(w_1,\dots,w_n)=v \text{ where } ((w_1,\dots,w_n),v) \leftarrow \langle \tilde{P}, V(\mathbb{F},H,n,\gamma,d)\rangle \right] \leq 1-\left(1-\frac{d}{|\mathbb{F}|}\right)^n \end{cases}$$

$P(\mathbb{F},H,n,\gamma,p)$ $\sum_{\alpha_1,\dots,\alpha_n \in H} p(\alpha_1,\dots,\alpha_n) \stackrel{?}{=} \gamma$ $V^{\blacksquare}(\mathbb{F},H,n,\gamma,d)$ } NO query access to $p$

| | | |
|---|---|---|
| $p_1(X) := \sum_{\alpha_2,\dots,\alpha_n \in H} p(X,\alpha_2,\dots,\alpha_n)$ | $\xrightarrow{p_1 \in \mathbb{F}[x]}$ | $\sum_{\alpha_1 \in H} p_1(\alpha_1) \stackrel{?}{=} \gamma$ |
| | $\xleftarrow{\omega_1 \in \mathbb{F}}$ | $\omega_1 \leftarrow \mathbb{F}$ |
| $p_2(X) := \sum_{\alpha_3,\dots,\alpha_n \in H} p(\omega_1,X,\alpha_3,\dots,\alpha_n)$ | $\xrightarrow{p_2 \in \mathbb{F}[x]}$ | $\sum_{\alpha_2 \in H} p_2(\alpha_2) \stackrel{?}{=} p_1(\omega_1)$ |
| | $\xleftarrow{\omega_2 \in \mathbb{F}}$ | $\omega_2 \leftarrow \mathbb{F}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $p_n(X) := p(\omega_1,\dots,\omega_{n-1},X)$ | $\xrightarrow{p_n \in \mathbb{F}[x]}$ | $\sum_{\alpha_n \in H} p_n(\alpha_n) \stackrel{?}{=} p_{n-1}(\omega_{n-1})$ |
| | $\xleftarrow{\omega_n \in \mathbb{F}}$ | $\omega_n \leftarrow \mathbb{F}$ |
| | | Output $((\omega_1,\dots,\omega_n), p_n(\omega_n))$ |

We use this perspective throughout the course.

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